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FREEZING OF FLUIDS IN FORCED FLOW

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Abstract—The freezing of a solid phase from a fluid flowing past a cold surface is analyzed. The time dependent, local solid layer thickness and the time dependent temperature distribution in the solid phase are determined as functions of the pertinent physical properties, the surface temperature and the surface convective heat flux. The latter may be a function of the streamwise coordinate. An exact, numerical, solution is obtained and is compared to the approximate solution of Lapadula and Mueller and two limiting solutions for very small and very large times. Numerical examples are given to illustrate the method of calculation.

NOMENCLATURE

C, specific heat of the solid phase;
H, dimensionless thickness parameter for solid phase;
k, thermal conductivity of the solid phase;
L, local steady-state thickness of the solid phase;
q_c, convective heat flux from fluid to solid phase;
$s(t)$, local thickness of the solid;
t, time;
T, temperature;
T_f, fusion temperature of the solid phase;
T_p, temperature of the cold surface;
x, dimensional coordinate tangent to the cold surface;
y, dimensional coordinate normal to the cold surface.

Greek symbols

α, thermal diffusivity of the solid phase;
γ, dimensionless physical parameter, $2k\Delta T/\rho\lambda\alpha$;
λ, heat of fusion;
η, dimensionless space variable;
θ, dimensionless temperature;
ρ, mass density of the solid phase;
τ, dimensionless time;
τ^*, dimensionless time at which limiting solution for large time is started;
ΔT, temperature difference, $T_f - T_p$.

INTRODUCTION

THE FREEZING of a liquid probably first received analytical treatment in Stefan's [1] now-classical work on formation of polar ice. Since, phenomena of freezing and melting of liquids and solids, as well as their analogues in the vapor and solid phases, have come to be recognized as comprising a class of "Stefan-like" problems.

In recent years, this class of problems has attracted attention in diverse quarters. Those works reviewed by the authors could be grouped in three broad categories. First, the exact closed-form solutions of Stefan, Neumann [2] and Rosenthal [3], which are available when certain restrictions are imposed. The second classification includes approximate solutions to problems

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of freezing and melting. Two basically different methods, associated with the works of Goodman [4, 5] and Biot [6, 7], are employed. Libby and Chen [8] applied Goodman's integral method to the growth of a solid layer deposited from a flowing gas, and Lapadula and Mueller [9] employed Biot's method in dealing with the same problem. The third category contains those works offering finite-difference solutions, among which those by Douglas [10] and Landau [11] are prominent.

Additionally, the electrical-network analogy of Kreith and Romie [12], and the series solution of Evans, Isaacson and MacDonald [13] should be mentioned. A brief but comprehensive survey of numerical and analytical methods was published in 1959 by Murray and Landis [14], who also introduced two new numerical procedures.

Here, a finite-difference technique will be employed to yield the thickness of the solid phase deposited by a flowing liquid on a cold surface, as a function of time and location on the surface.

ANALYSIS

The problem to be considered is that of freezing of a fluid in steady plane flow over a cold surface.

The basic assumptions employed are:

a. Thickness of the deposited layer, s, is sufficiently small that conduction of heat in the layer may be considered spatially one-dimensional.
b. The convective heat flux, q_c, transferred from the fluid to the cold surface is known as a function of the streamwise coordinate, x, and is time-independent.
c. The physical properties of both fluid and solid phases are constant.
d. The temperature of the cold surface, T_p, is uniform and constant.
e. It is assumed that there exists a definite interface between the fluid and solid phases.

Figure 1 shows the physical system and some of the notation employed.

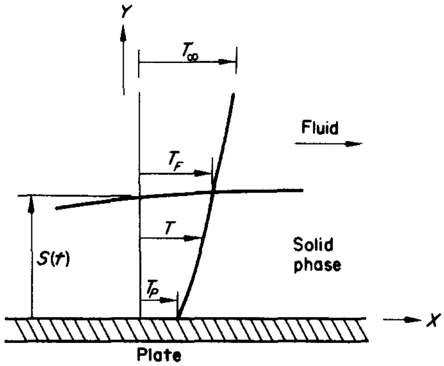

FIG. 1. The physical system.

The equations governing the flow of heat in and growth of the solid phase are:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{1}$$

$$q_c + \rho\lambda \frac{\mathrm{d}s}{\mathrm{d}t} = k \frac{\partial T}{\partial y}(s, t). \tag{2}$$

Boundary and initial conditions are:

$$\left.\begin{array}{ll} y = 0, & T = T_p \\ y = s, & T = T_f \\ t = 0, & s = 0. \end{array}\right\} \tag{3}$$

Equation (1) is the familiar one-dimensional heat-conduction equation. Equation (2) expresses a balance between the flux of heat to the solid from the fluid and the flux of heat conducted away from the interface in the solid.

The problem is transformed, for convenience, into a non-dimensional space. The transforming equations are:

$$\eta = \frac{Y}{s(t)} \tag{4}$$

$$\theta = \frac{T - T_p}{T_f - T_p} \tag{5}$$

$$\tau = \frac{\alpha t}{L^2}. \tag{6}$$

In equation (6) the characteristic length, L, used to non-dimensionalize time is the local steady-state thickness approached by the solid layer as $t \to \infty$:

$$L = \frac{k(T_f - T_p)}{q_c}. \tag{7}$$

It is through this parameter that the specified heat flux, q_c, enters the non-dimensional analysis.

Upon application of transformations (4–6), equations (1–3) become the following:

$$H(\tau)\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\eta^2} + \frac{\eta}{2}\frac{dH}{d\tau}\frac{\partial\theta}{\partial\eta} \tag{8}$$

$$\frac{dH}{d\tau} = \frac{2k\Delta T}{\rho\lambda\alpha}\left[\frac{\partial\theta}{\partial\eta}(1,\tau) - (H)^{\frac{1}{2}}\right] \tag{9}$$

$$\left.\begin{aligned} \theta(0,\tau) &= 0 \\ \theta(1,\tau) &= 1 \\ H(0) &= 0 \end{aligned}\right\} \tag{10}$$

where H is a non-dimensional solid thickness:

$$H = \left(\frac{s(t)}{L}\right)^2. \tag{11}$$

LIMITING SOLUTIONS

Two observations pertinent in the neighborhood of $\tau = 0$ may be made:

a. Because $H(0) = 0$, $(H)^{\frac{1}{2}}$ is near zero for very small values of τ, while $dH/d\tau$ and $\partial\theta/\partial\eta\,(1,\tau)$ are not small.

b. θ assumes constant values on the boundaries, and this suggests that θ may depend only on η, near $\tau = 0$.

Assuming, then, that

$$\theta = F(\eta) \tag{12}$$

and simplifying equation (9) with $(H)^{\frac{1}{2}} \approx 0$, the following limiting solutions may be obtained, for τ near zero:

$$\theta(\eta) = \frac{\int_0^\eta \exp(-bz^2)\,dz}{\int_0^1 \exp(-bz^2)\,dz} \tag{13}$$

$$H(\tau) = 4b\tau. \tag{14}$$

The constant b is determined by the relationship

$$(\sqrt{b})\exp(b)\,\mathrm{erf}(\sqrt{b}) = \frac{k\Delta T}{\rho\lambda\alpha\sqrt{\pi}}. \tag{15}$$

In order to obtain a limiting solution valid for large values of τ, it may be pointed out here that results of numerical integration (summarized below) show that $\theta(\eta, \tau)$ approaches its steady-state distribution much more rapidly than does $H(\tau)$. Therefore, it may be assumed that for τ greater than some large value, say τ^*, the temperature gradient at $\eta = 1$ is

$$\frac{\partial\theta}{\partial\eta}(1,\tau) = 1. \tag{16}$$

Further, let $H(\tau)$ be represented by

$$H(\tau) = H(\infty) - \bar{H}(\tau) = 1 - \bar{H}. \tag{17}$$

These substitutions make possible the limiting solution:

$$\tau = \tau^* + \frac{\rho\lambda\alpha}{k\Delta T}\left[(H^*)^{\frac{1}{2}} - (1-\bar{H})^{\frac{1}{2}} - \ln\frac{1-(1-\bar{H})^{\frac{1}{2}}}{1-(H^*)^{\frac{1}{2}}}\right] \tag{18}$$

valid for large τ. The notation, $H^* = H(\tau^*)$ is used.

NUMERICAL SOLUTION

Exact, closed-form solutions of equations (8–10) have been obtained only with the aid of rather restrictive assumptions concerning the convective heating of the solid layer by the fluid. These restrictions on q_c are reflected in the limitations to large or small values of τ [see equations (6) and (7)].

In this work, a modified Crank–Nicholson finite-difference representation was substituted for equation (8), and a modified fourth-order Runge–Kutta procedure was used to integrate equation (9) for the general case of a known non-zero, time-independent convective heating rate.

Equations (8) and (9) are mutually coupled and equation (9) is non-linear. Therefore, an iterative procedure was included in the integration which gives H and the distribution of θ at the jth time level, using successive determinations of the θ-distribution at the $(j + 1)$th time-level to refine the calculated value of H at that time level. The iteration was continued at each time-level until the dimensionless temperature gradient, $\partial\theta/\partial\eta$, evaluated at the solid–fluid interface, $\eta = 1$, varied by less than 10^{-4} from one iteration to the next. The value of H at the $(j + 1)$th time level was then calculated using this "latest" value of the temperature gradient.

The finite-difference equations used, along with additional discussion of the numerical procedures, are given by Beaubouef [15].

Results of the numerical integration, for values of $2k\,\Delta T/\rho\lambda\alpha = 0{\cdot}10$, $1{\cdot}0$ and $10{\cdot}0$ are given in Figs. 2, 3 and 4, respectively.

RESULTS

The numerical solutions for $\gamma = 0{\cdot}1$, $1{\cdot}0$ and $10{\cdot}0$ are shown in Figs 2, 3 and 4, respectively, where they are compared to the limiting solutions and to the approximate solutions of Lapadula and Mueller (for $\pi = 0$; see [9]).

It may be observed that the numerical solutions, in all cases, show the proper limiting behavior. Further, Figs. 2–4 demonstrate generally good agreement between the approximate solutions of [9] and the numerical solutions of this work.

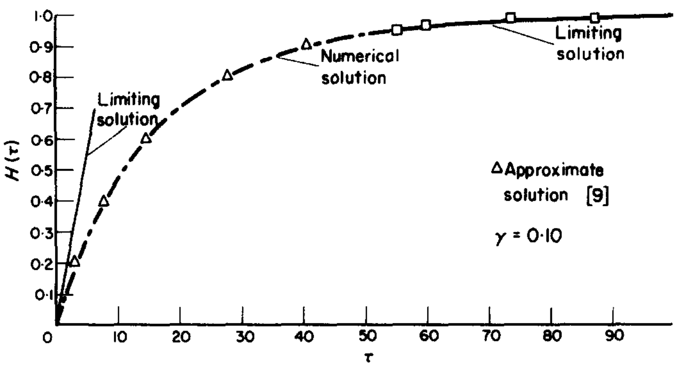

FIG. 2. Comparison of the exact, approximate and limiting solutions for $\gamma = 0{\cdot}1$.

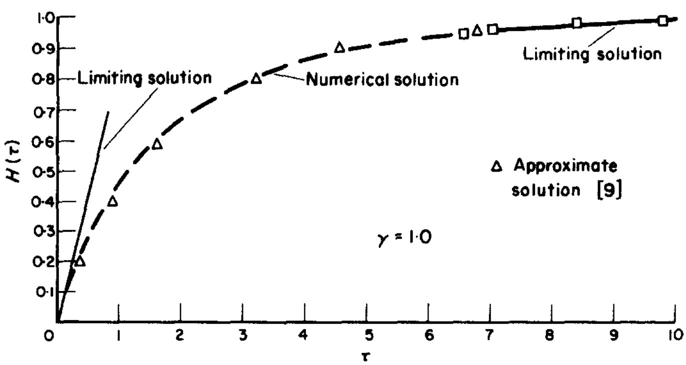

FIG. 3. Comparison of the exact, approximate and limiting solutions for $\gamma = 1{\cdot}0$.

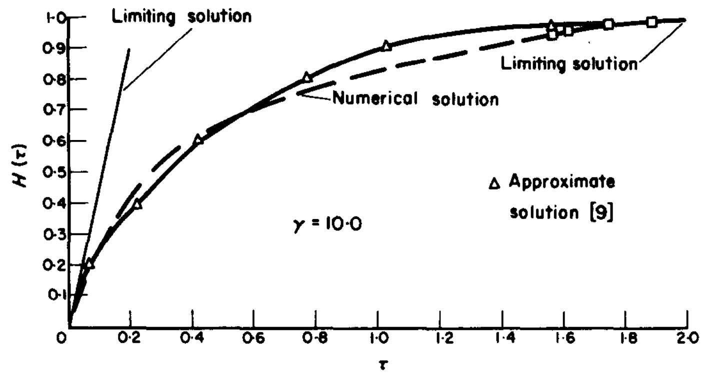

FIG. 4. Comparison of the exact, approximate and limiting solutions for $\gamma = 10{\cdot}0$.

These results demonstrate the marked influence of physical properties, boundary conditions and convective heating rate (via the parameter L in τ) on the freezing rate of a flowing fluid. Table 1, showing the time when $H = 0{\cdot}99$ for the values of γ employed, illustrates this point. As γ is increased through two orders of magnitude, the time required for the freezing transient decreases by two orders of magnitude.

Table 1

γ	τ for $H = 0{\cdot}99$
0·1	86
1·0	10
10·0	2

It should be noted that

$$\gamma = \frac{2k\,\Delta T}{\rho\lambda\alpha} = \frac{2C\,\Delta T}{\lambda},$$

and appears as a "sensible heat/latent heat" ratio. Materials with small latent heat, λ, will be characterized by large γ; because less latent heat must be extracted in order to freeze a unit mass, such materials should freeze more rapidly. The converse is true for materials with large latent heat.

Additionally, materials with small specific heat, C, in the solid phase are characterized by small γ. Such materials should demonstrate transients in θ which are negligible compared to the transients in H, and conversely.

NUMERICAL EXAMPLES

Results of the numerical integration for the case $\gamma = 0{\cdot}10$ will be employed in calculating the thickness, $s(t)$, of the ice layer deposited from a flowing water stream in the two cases of: (a) plane stagnation flow; and (b) flow over a flat plate at zero angle of incidence. The physical properties of water–ice, at atmospheric pressure, will be taken to be

$$k = 1{\cdot}28 \text{ Btu/(h-ft-°F)}$$
$$C = 0{\cdot}49 \text{ Btu/(lb-°F)}$$
$$\lambda = 144 \text{ Btu/lb}$$
$$\alpha = 0{\cdot}046 \text{ (ft)}^2\text{/h}.$$

Case (a): Plane stagnation flow

The convective heat flux, q_c, in steady laminar plane stagnation flow is independent of the streamwise coordinate, x. Therefore, the thickness of the deposited solid phase will be uniform over the "plate" at each instant. For an example, the convective heat flux is chosen to be

$$q_c = 500 \text{ Btu/(h-ft}^2\text{)}.$$

It follows, then, that:

$$\Delta T = 14{\cdot}8 \text{ degF (chosen so that } \gamma = 0{\cdot}10)$$
$$L = 0{\cdot}0379 \text{ ft}$$
$$s(\tau) = 0{\cdot}0379\,(H)^{\frac{1}{2}} \text{ ft}$$
$$t = 0{\cdot}0313\,(\tau) \text{ h.}$$

Table 2 shows corresponding values of $s(t)$ and t for this case. From this table, it may be seen that the example system would "reach" steady-state in just over 2·58 h.

Table 2. Growth of ice layer in stagnation flow, case (a)

t (h)	$s(t)$, (ft)
0·0	0·0
0·04989	0·0130
0·1745	0·0216
0·2990	0·0259
0·5500	0·0308
0·8600	0·0339
0·1700	0·0355
1·4900	0·0365
2·5800	0·0378

Case (b): Flow over a flat plate at zero incidence

It is assumed that the presence of the solid phase deposited on the plate has negligible effect on the fluid flow; this is consistent with the assumption made earlier that the solid layer is thin. The variation of q_c with x will therefore be taken to be that for the flat plate alone. As is well known (see Schlichting [16], for example), in such a flow

$$q_c \approx \frac{1}{\sqrt{x}}.$$

For these calculations, the convective heat flux distribution is chosen to be

$$q_c = \frac{500 \text{ Btu/(h-ft}^2)}{\sqrt{x}}.$$

The length of the plate will be taken to be 1 ft. Thus,

$$L = 0{\cdot}0379\,(x)^{\frac{1}{2}} \text{ ft}$$
$$s(x, \tau) = 0{\cdot}0379\,(H)^{\frac{1}{2}} \text{ ft}$$
$$t = 0{\cdot}0313\,(\tau) \text{ h.}$$

The configuration of the solid–fluid interface was calculated at two times during the transient, ($t = 0{\cdot}299$ and $t = 1{\cdot}49$ h), and at the steady-state, for $x = 0, 0{\cdot}25, 0{\cdot}50, 0{\cdot}75, 1{\cdot}0$ ft. The results are given in Table 3.

Table 3. Growth of ice layer for flow past a flat plate, case (b)

x (ft)	t (h)	τ	$H(\tau)$	$s(x, t)$ (ft)
0·0	0·299	9·56	0·470	0·0
0·25	0·299	9·56	0·470	0·130
0·50	0·299	9·56	0·470	0·0184
0·75	0·299	9·56	0·470	0·0226
1·0	0·299	9·56	0·470	0·0260
0·0	1·49	47·56	0·928	0·0
0·25	1·49	47·56	0·928	0·0182
0·50	1·49	47·56	0·928	0·0258
0·75	1·49	47·56	0·928	0·0315
1·0	1·49	47·56	0·928	0·0364
0·0	steady state	steady state	1·0	0·0
0·25	steady state	steady state	1·0	0·0189
0·50	steady state	steady state	1·0	0·0268
0·75	steady state	steady state	1·0	0·0328
1·0	steady state	steady state	1·0	0·0379

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Résumé—La congélation d'un fluide s'écoulant le long d'une surface froide est analysée. L'épaisseur locale de la couche solide dépendant du temps et la distribution de température dépendant du temps dans la phase solid sont déterminées en fonction des propriétés physiques convenables, de la température de la surface et du flux de chaleur par convection à la surface. Ce dernier peut être une fonction de la coordonnée longitudinale. Une solution exacte et numérique est obtenue et comparée à la solution approchée de Lapadula et Mueller et à deux solutions limites correspondant à des temps très faibles ou très grands. On donne des exemples numériques pour illustrer la méthode de calcul.

Zusammenfassung—Das Ausfrieren einer festen Phase aus einer Flüssigkeit, die entlang einer kalten Oberfläche fliesst, wird analysiert. Die zeitabhängige, örtliche Dicke der festen Schicht und die zeitabhängige Temperaturverteilung in der festen Phase wird als Funktion der entsprechenden Stoffwerte, der Oberflächentemperatur und der konvektiven Wärmeabgabe an der Oberfläche bestimmt. Letztere kann eine Funktion der Koordinate in Strömungsrichtung sein. Eine exakte numerische Lösung wurde erhalten und dann mit der Näherungslösung von Lapadula und Mueller und zwei Grenzlösungen für sehr kleine und sehr grosse Zeiten verglichen. Zur Erklärung der Rechenmethode werden numerische Beispiele angegeben.

Аннотация—Анализируется вымерзание твердой фазы из жидкости, омывающей холодную поверхность. Зависящие от времени локальная толщина твердого слоя и распределение температуры в твердой фазе определяются как функции соответствующих физических свойств, температуры поверхности и конвективного теплового потока на поверхности. Последний можно выразить в виде функции продольной координаты. Полученное точное численное решение сравнивалось с приближенным решением Лападулы и Мюллера, а также с двумя предельными решениями для очень малых и очень больших значений времени. Приводятся численные примеры для иллюстрации метода расчета.